JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 2, March–April 1994

Identification of Nonlinear Interactions in Structures

B. Balachandran* and A. H. Nayfeh†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
S. W. Smith‡
University of Kentucky, Lexington, Kentucky 40506
and
R. S. Pappa§
NASA Langley Research Center, Hampton, Virginia 23665

Recent experiences in phenomenological identification of nonlinear interactions in structures are presented. Free oscillations of quadratically and cubically coupled pairs of oscillators are analytically studied to illustrate nonlinear interactions between structural modes involved in two-to-one and one-to-one frequency relationships. In light of this analytical study, results obtained from application of the Eigensystem Realization Algorithm (ERA) toward identification of a beam-mass structure with a two-to-one frequency relationship and quadratic coupling are presented and discussed. The internal resonance in the structure causes the identified damping coefficients to be oscillatory and to assume negative values. The present effort should be of relevance to the identification of structural systems in which factors, such as symmetry, modal density, frequency relationships, and flexibility contribute to the likelihood that nonlinear interactions may occur.

I. Introduction

STRUCTURAL identification is of paramount importance for the prediction of the response of a given structure to various loading conditions. The dynamic characteristics of a structure that exhibits nonlinear behavior can be qualitatively different for different loading conditions. Although the responses of many structures used in practice can be determined by modeling them with linear equations, there are many physical phenomena that cannot be modeled by linear equations.[1,2] In many structures, some of the natural frequencies are commensurate or nearly commensurate. In the presence of appropriate nonlinearities, these frequency relationships can lead to nonlinear interactions. In this event, such frequency relationships are called internal resonances. Nonlinear interactions can occur during both free and forced oscillations of a structure and lead to energy exchange or energy transfer between the coupled modes. With the advent of complex space and aircraft structures, linear models with thousands of degrees of freedom are not uncommon. To identify the parameters in the linear models, techniques including the ERA method[3–5] are employed. Because conditions for nonlinear phenomena do exist in most of such structures (see, e.g., Refs. 5 and 6), there is a growing need for the identification of nonlinear systems. Many studies have addressed nonlinear identification (see, e.g., Refs. 7–9) for systems with one or a few degrees of freedom. However, these studies have not allowed for nonlinear interactions. Identification of systems that exhibit nonlinear interactions is just beginning to receive attention,[10,11] again for less complex structures. It is therefore of interest to examine if the phenomenological identification of nonlinear interactions is possible within the context of complex structures and linear system identification techniques.

Presented as Paper 92-0491 at the AIAA Aerospace Sciences Meeting, Reno, NV, Jan. 6–9, 1992; received Aug. 15, 1992; revision received May 24, 1993; accepted for publication July 20, 1993.

*Research Associate, Department of Engineering Science and Mechanics; currently Assistant Professor, Department of Mechanical Engineering, University of Maryland, College Park, MD.

†University Distinguished Professor, Department of Engineering Science and Mechanics. Fellow AIAA.

‡Assistant Professor, Department of Engineering Mechanics. Member AIAA.

§Aerospace Engineer, Structural Dynamics Division. Member AIAA.

The purpose of the present work is to show that phenomenological identification of nonlinear interactions may be carried out by using linear system identification techniques. The material of this paper is organized as follows. In Sec. II, we consider free oscillations of quadratically and cubically coupled pairs of oscillators to illustrate nonlinear interactions. In the case of the quadratically coupled oscillators, there is a two-to-one frequency relationship between the two modes of oscillation. On the other hand, in the case of the cubically coupled oscillators, there is a one-to-one frequency relationship between the two modes of oscillation. Equations derived by the method of multiple scales,[1] which govern the time variation of the amplitudes and phases of the interacting modes, are studied in Sec. II. Subsequently, we present results of application of ERA to free-oscillation data obtained from an L-shaped, beam-mass structure. The second natural frequency f_2 of this structure is approximately twice the first natural frequency f_1. These frequencies are less than 20 Hz while the other frequencies are above 100 Hz. The first and second modes are coupled through inertial quadratic nonlinearities, which lead to a two-to-one internal resonance and consequent modal interactions in this and similar structures.[12–15] In Sec. IV, we discuss the experimental results obtained in the Mini-Mast experiments[5] in the context of the present work. Finally, we close the paper with some concluding remarks.

II. Analysis

In this section, we consider free oscillations of two sets of coupled oscillators. In both sets the linear part of the damping is light and the nonlinear terms are weak; thus, the weakly damped and weakly nonlinear systems are amenable to perturbation analyses. It may be necessary to use numerical methods for studying equations with strong nonlinear terms.

A. Quadratically Coupled Oscillators

We consider a pair of quadratically coupled oscillators with a two-to-one frequency relationship between the two modes of oscillation. The governing equations are

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \mu_3 \dot{u}_1^2 \operatorname{sgn}|\dot{u}_1| + \alpha_1 u_1 u_2 = 0 \tag{1}$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \mu_4 \dot{u}_2^2 \operatorname{sgn}|\dot{u}_2| + \alpha_2 u_1^2 = 0 \tag{2}$$

$$\mu i = \varepsilon \hat{\mu}_i, \quad \alpha_i = \varepsilon \hat{\alpha}_i, \quad \omega_2 = 2\omega_1 + \sigma, \quad \sigma = \varepsilon \hat{\sigma} \tag{3}$$

where the u_i, ω_i, μ_i, α_i, and σ are, respectively, the generalized coordinates, the circular frequencies, the damping coefficients, the nonlinear coefficients, and the internal detuning. The overdots denote derivatives with respect to time t. The parameter ε is small and positive; it is used to explicitly express our assumption that the damping and nonlinearities are weak. Also note that Eqs. (1) and (2) have quadratic damping terms. The two modes of oscillation are linearly uncoupled but nonlinearly coupled through the quadratic terms. In the context of a structure, Eqs. (1) and (2) can describe the modal amplitudes of two quadratically coupled structural modes.

Using the method of multiple scales,[1] we obtain a first approximation for the solution of Eqs. (1) and (2) as

$$u_1 \approx p_1 \cos \omega_1 t + q_1 \sin \omega_1 t \tag{4}$$

$$u_2 \approx p_2 \cos \omega_2 t + q_2 \sin \omega_2 t \tag{5}$$

The p_i and q_i are governed by the following equations:

$$\dot{p}_1 = -\mu_1 p_1 - \frac{4\mu_3}{3\pi}\left(p_1\sqrt{p_1^2+q_1^2}\right) - \frac{1}{2}\sigma q_1 - \Lambda_1 (p_2 q_1 - p_1 q_2) \tag{6}$$

$$\dot{q}_1 = -\mu_1 q_1 - \frac{4\mu_3}{3\pi}\left(q_1\sqrt{p_1^2+q_1^2}\right) + \frac{1}{2}\sigma p_1 - \Lambda_1 (p_1 p_2 + q_1 q_2) \tag{7}$$

$$\dot{p}_2 = -\mu_2 p_2 - \frac{4\mu_4}{3\pi}\left(p_2\sqrt{p_2^2+q_2^2}\right) + 2\Lambda_2 p_1 q_1 \tag{8}$$

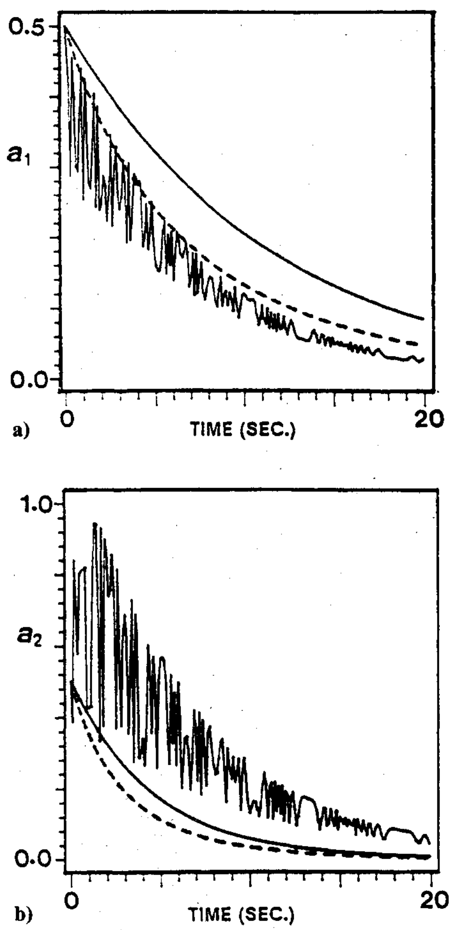


Fig. 1 **Free oscillation in the absence and presence of quadratic coupling terms: a) first mode and b) second mode.**

$$\dot{q}_2 = -\mu_2 q_2 - \frac{4\mu_4}{3\pi}\left(q_2\sqrt{p_2^2+q_2^2}\right) - \Lambda_2 (p_1^2 - q_1^2) \tag{9}$$

where the Λ_i are given by

$$4\omega_1\Lambda_1 = \alpha_1 \text{ and } 4\omega_2\Lambda_2 = \alpha_2 \tag{10}$$

The Λ_i should have the same sign in the case of structures.[1] This condition ensures that, in the presence of positive damping, structures do not exhibit self-sustained oscillations. In arriving at Eqs. (6–9), we had assumed that the p_j and q_j are different from zero.

Equations (6–9) are numerically integrated to determine the modal amplitudes $a_j = \sqrt{p_j^2 + q_j^2}$. In Fig. 1, we display the modal amplitudes corresponding to the free oscillations initiated from $(p_1, q_1, p_2, q_2) = (0.4, 0.3, 0.4, 0.3)$. For all the results of Fig. 1, $\mu_1 = 0.09$, $\mu_2 = 0.22$, and $\sigma = 1.13$. The solid lines that depict monotonic decay correspond to the case where only the linear damping terms are present; that is, $\mu_3 = 0$, $\mu_4 = 0$, and $\alpha_i = 0$. The broken lines in Figs. 1a and 1b that depict monotonic decay correspond to the case where both the linear and quadratic damping terms are present. In this case, $\mu_3 = 0.4$, $\mu_4 = 0.8$, and $\alpha_i = 0$. Hence, in the absence of nonlinear coupling terms, the free decay is always monotonic. The solid lines that depict nonmonotonic decay correspond to the case where the nonlinear coefficients assume values different from zero. Here, $\mu_3 = \mu_4 = 0$, $\Lambda_1 = 293.613$, and $\Lambda_2 = 2208.297$. The parameter values correspond to the lightly damped, L-shaped, beam-mass structure treated in Ref. 14. The free decay of the modal amplitudes is oscillatory as a consequence of the energy exchange between the first and second modes. It is evident that as a_1 increases a_2 decreases and vice versa. Also note that the amplitude of the second mode grows initially before falling off. The period of oscillation depends on the initial conditions, the detuning parameter, and the nonlinear coefficients. The free decay was also seen to be oscillatory in the presence of quadratic damping terms.

B. Cubically Coupled Oscillators

In this section, we consider a pair of cubically coupled oscillators with a one-to-one frequency relationship between the two modes of oscillation. The oscillators are governed by

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1\dot{u}_1 + \mu_3\dot{u}_1^3 + \alpha_1 u_1^3 + \alpha_2 u_2^2 u_1 = 0 \tag{11}$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2\dot{u}_2 + \mu_4\dot{u}_2^3 + \alpha_3 u_2^3 + \alpha_4 u_1^2 u_2 = 0 \tag{12}$$

$$\mu_i = \varepsilon\hat{\mu}_i,\ \alpha_i = \varepsilon\hat{\alpha}_i,\ \omega_2 = \omega_1 + \sigma,\ \sigma = \varepsilon\hat{\sigma} \tag{13}$$

where the u_i, μ_i, α_i, and ω_i are as defined earlier. Equations (11) and (12) have cubic damping terms. We consider cubic damping terms to illustrate the effect of higher-order damping. These equations can describe the modal amplitudes of two cubically coupled structural modes. In the case of a structure, the coefficients α_2 and α_4 have the same sign, and hence, in the presence of positive damping, the structure does not exhibit self-sustained oscillations.

One-to-one frequency relationships are possible in systems with symmetry, such as the Mini-Mast structure.[5] Again, using the method of multiple scales, we obtain the first approximation (4) and (5). However, in this case, the p_j and q_j are governed by

$$\begin{aligned} \dot{p}_1 = {} & -\mu_1 p_1 - \sigma q_1 - \frac{3\omega_1^2\mu_3}{8}(p_1^2+q_1^2)\,p_1 \\ & + \frac{3\alpha_1}{8\omega_1}(p_1^2+q_1^2)\,q_1 + \frac{2\alpha_2}{8\omega_1}(p_2^2+q_2^2)\,q_1 \\ & - \frac{\alpha_2}{8\omega_1}(p_2^2-q_2^2)\,q_1 + \frac{\alpha_2}{8\omega_1}(2p_2q_2p_1) \end{aligned} \tag{14}$$

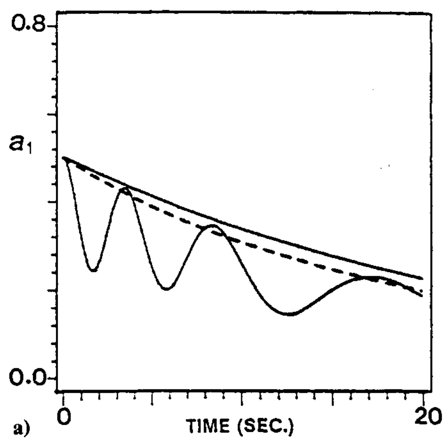
$$\dot{q}_1 = -\mu_1 q_1 + \sigma p_1 - \frac{3\omega_1^2\mu_3}{8}(p_1^2+q_1^2)q_1 - \frac{3\alpha_1}{8\omega_1}(p_1^2+q_1^2)p_1 + \frac{2\alpha_2}{8\omega_1}(p_2^2+q_2^2)p_1 - \frac{\alpha_2}{8\omega_1}(p_2^2-q_2^2)p_1 - \frac{\alpha_2}{8\omega_1}(2p_2q_2q_1) \tag{15}$$

$$\dot{p}_2 = -\mu_2 p_2 - \frac{3\omega_2^2\mu_4}{8}(p_2^2+q_2^2)p_2 + \frac{3\alpha_3}{8\omega_2}(p_2^2+q_2^2)q_2 + \frac{2\alpha_4}{8\omega_2}(p_1^2+q_1^2)q_2 - \frac{\alpha_4}{8\omega_2}(p_1^2-q_1^2)q_2 + \frac{\alpha_4}{8\omega_2}(2p_1q_1p_2) \tag{16}$$

$$\dot{q}_2 = -\mu_2 q_2 - \frac{3\omega_2^2\mu_4}{8}(p_2^2+q_2^2)q_2 - \frac{3\alpha_3}{8\omega_2}(p_2^2+q_2^2)p_2 - \frac{2\alpha_4}{8\omega_2}(p_1^2+q_1^2)p_2 - \frac{\alpha_4}{8\omega_2}(p_1^2-q_1^2)p_2 - \frac{\alpha_4}{8\omega_2}(2p_1q_1q_2) \tag{17}$$

Let

$$\mu_{3h} = \frac{3}{8}\omega_1^2\mu_3,\quad \mu_{4h} = \frac{3}{8}\omega_2^2\mu_3,\quad \Gamma_1 = \frac{3\alpha_1}{8\omega_1} \tag{18}$$



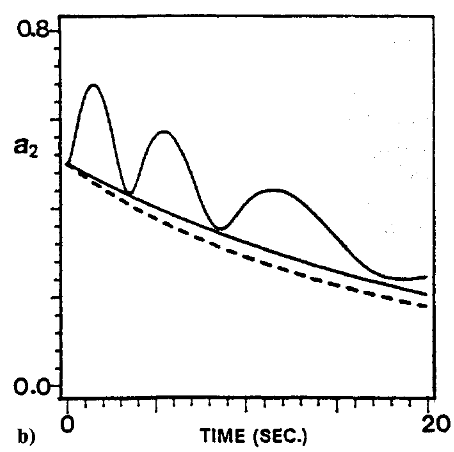


Fig. 2 Free oscillation in the absence and presence of cubic coupling terms: a) first mode and b) second mode.

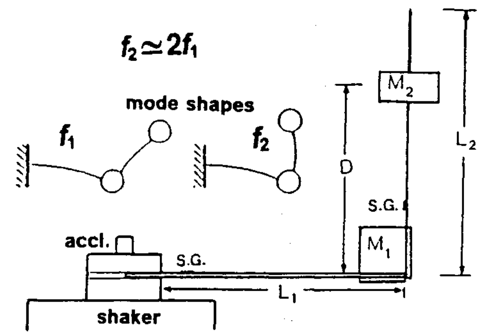


Fig. 3 Metallic structure and accompanying shapes of the first two modes of vibration; the second linear natural frequency f_2 is approximately twice the first linear natural frequency f_1.

and

$$\Gamma_2 = \frac{\alpha_2}{8\omega_1},\quad \Gamma_3 = \frac{3\alpha_3}{8\omega_2},\quad \Gamma_4 = \frac{\alpha_4}{8\omega_2} \tag{19}$$

Equations (14–17) are numerically integrated to determine the modal amplitudes $a_j = \sqrt{p_j^2+q_j^2}$. In Fig. 2, we display the modal amplitudes corresponding to the free oscillation initiated from (p_1, q_1, p_2, q_2) = (0.3, 0.4, 0.3, 0.4). For all the results of Fig. 2, μ_1 = 0.04, μ_2 = 0.045, and σ = 0.05. The solid lines in Figs. 2a and 2b that depict monotonic decay correspond to the case where $\mu_{3h} = 0$, $\mu_{4h} = 0$, and $\Gamma_i = 0$. The broken lines that depict monotonic decay correspond to the case where $\mu_{3h} = 0.06$, $\mu_{4h} = 0.07$, and $\Gamma_i = 0$. Hence, in the absence of nonlinear coupling terms, the free decay is always monotonic. The solid lines that depict nonmonotonic decay correspond to the parameter values $\mu_{3h} = 0$, $\mu_{4h} = 0$, Γ_1 = 2.0, Γ_2 = 2.0, Γ_3 = 3.0, and Γ_4 = 3.0. On examining Fig. 2b, we find that the amplitude of the second mode initially grows before falling off. We also note that a_1 increases a_2 decreases and vice versa. The decay is oscillatory as a consequence of the energy exchange between the first and second modes.

From this section, it should be clear that the free decay is always monotonic in the presence of only damping and nonmonotonic in the presence of nonlinear interactions. We note that nonlinear interactions may occur in the presence of other frequency relationships, such as $\omega_i \cong 3\omega_j$ and $\omega_i \cong |\omega_j \pm \omega_k|$, where the ω_i, ω_j, and ω_k are the circular natural frequencies of a structure. We should also mention that, depending on the structure, it may be necessary to include other types of damping, such as Coulomb damping, hysteretic damping, and displacement-dependent damping in the analytical models described in Sec. II.

III. Experiments

The beam-mass structure used in the experiments is shown in Fig. 3. It consists of two lightweight steel beams and two rigid masses. One of the ends of the L-shaped structure is clamped to a mount. The mass M_2 is positioned to tune the frequencies of the structure. The mass M_1 weighs 31.1 g, and the mass M_2 weighs 15.5 g. The horizontal beam is 12.83 mm wide and 1.68 mm thick, and the vertical beam is 12.80 mm wide and 0.56 mm thick. With respect to Fig. 3, L_1, L_2, and D are 145.05, 152.40, and 124.02 mm, respectively. The structure was instrumented with strain gauges on the horizontal and vertical beams to obtain a measure of the response.

Using random excitations and nonstationary sinusoidal sweeps as described in Ref. 14, we determined the first and second natural frequencies of the structure. The first natural frequency f_1 is about 8.20 Hz, and the second natural frequency f_2 is about 16.72 Hz. Due to the presence of quadratic coupling between the first and second modes, the spectrum of the free response of the structure initially consists of spectral lines at f_1, $f_2/2$, f_2, and $2f_1$ in a baseband of 20 Hz. Sufficient resolution is required to resolve these

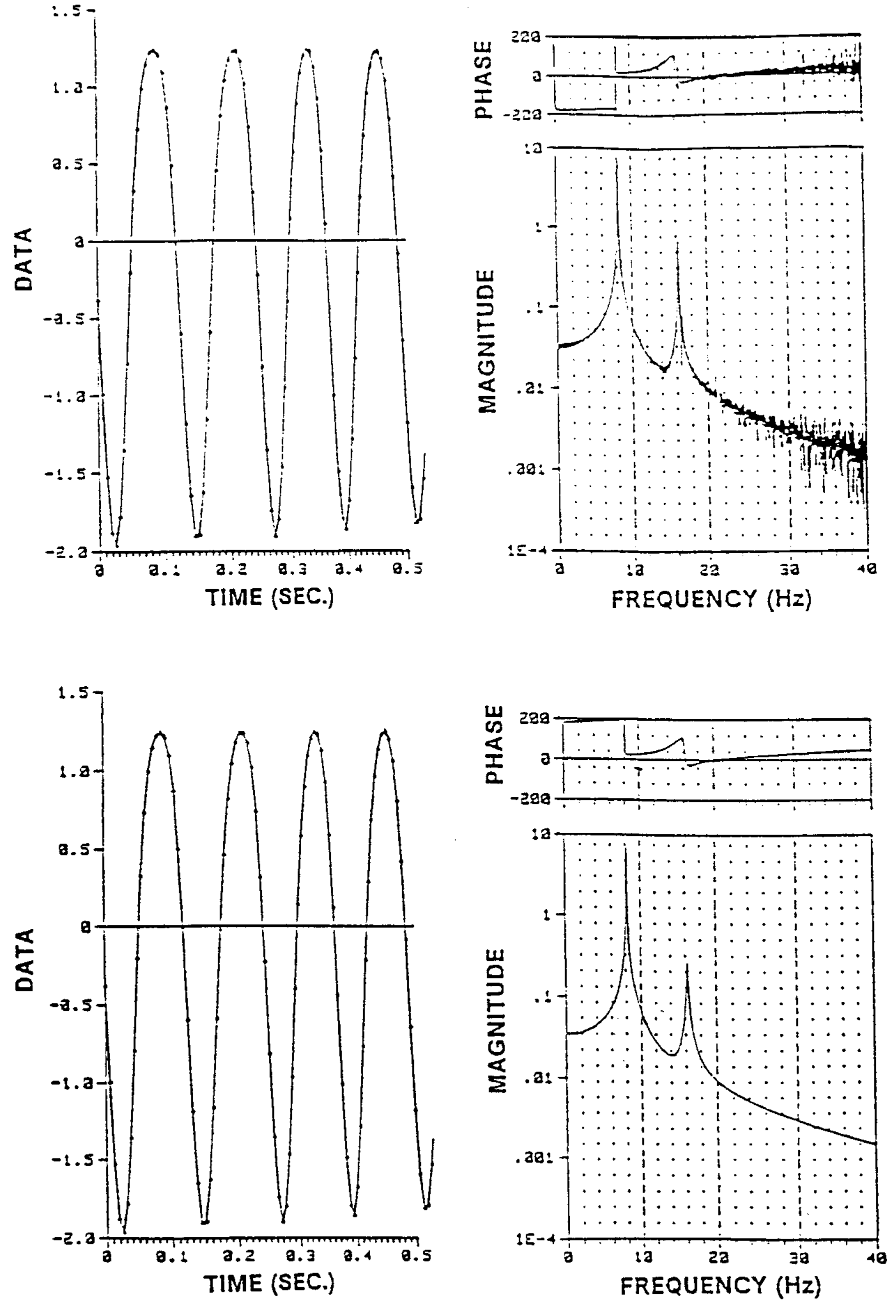


Fig. 4 Experimental data compared to linear model reconstructed from ERA results.

peaks. The nonlinear dynamics of harmonically excited L-shaped, beam-mass structures with a two-to-one frequency relationship are well documented in Refs. 12–15. Their dynamic behavior is very sensitive to the detuning (i.e., f_2–$2f_1$) and some of their nonlinear motions cannot be predicted if the correct frequencies are not identified. For example, the structure used in this study displayed two-period, quasiperiodic motions when excited by a harmonic excitation at 16.55 Hz. These motions cannot be analytically predicted if the identified second natural frequency is less than 16.55 Hz.

The structure was initially excited with a band-limited random excitation for a few seconds. Then the excitation was turned off before collecting a signal from the strain gauge on the vertical beam during free oscillations of the structure. The record is 32 s long and was collected at a sampling frequency of 160 Hz. This record was analyzed by using the ERA method. This algorithm, based on linear analysis in the time domain, is used to identify the modal parameters of a structure from input/output data. Further, it is a minimum-realization algorithm; that is, it identifies the minimum number of modes required to represent a structure. Applications of ERA to structural systems which exhibit nonlinear behavior have been reported in Refs. 4 and 5. However, the nonlinear dynamics of the complex structural systems of Refs. 4 and 5 are not well understood. Here we apply ERA to a beam-mass structure whose nonlinear dynamics are fairly well understood.

To identify the modal characteristics of a linear multivariable system by using ERA, sets of either impulse response functions or free-decay data are required, although a recent related algorithm[16] enables the identification to be carried out from sets of general input/output data. When ERA is applied to blocks of data of a free response of a structure (sliding-window analysis), it can capture the dependence of the natural frequency on the amplitude of oscillation (a nonlinear phenomenon[1]) provided that the mode in question does not interact with other modes.

From the free-response data, ERA identified two modes with high confidence in the frequency bandwidth of 0–40 Hz. Further, the first natural frequency was identified to be about 8.14 Hz and the second natural frequency was identified to be about 16.30 Hz. The second harmonic $2f_1$ of f_1 was identified to be the second natu-

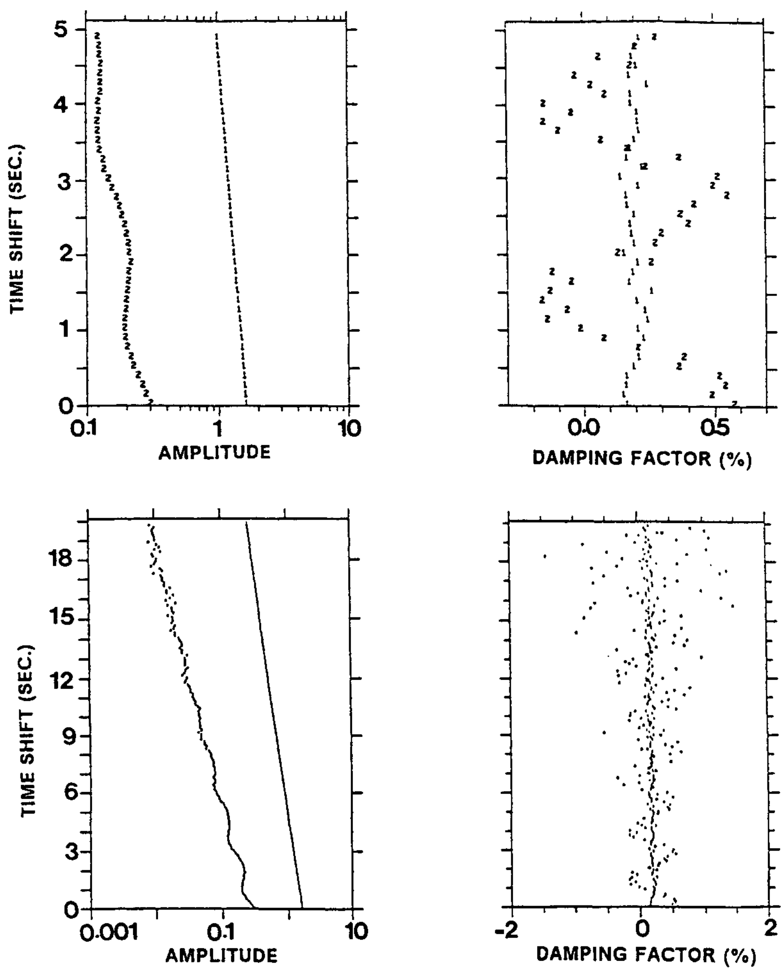


Fig. 5 ERA sliding-window results for the beam-mass structure.

ral frequency. This is believed to be a consequence of insufficient resolution. In Fig. 4, the figures in the upper half correspond to the data collected from the experiment and the figures in the lower half correspond to the linear model reconstructed from the ERA results. Some differences in the phases and magnitudes can be discerned. There seems to be a good overall agreement between the experimental and ERA results. However, a nonlinear analytical model constructed with the identified frequencies 8.14 and 16.30 Hz will not be able to predict the modulated motions observed at the excitation frequency 16.55 Hz.

In Fig. 5, we display the ERA results obtained from a sliding-window analysis. The figures in the upper half pertain to the first five seconds of the free-oscillation data while the figures in the lower half pertain to the first twenty seconds of the data. In the figures, the points labeled 1 (2) correspond to the first (second) mode. Variations of the logarithm of the amplitudes and damping factors with time are plotted in each set of figures. We observe oscillations in the modal amplitudes and damping factors. Further, the damping factor associated with the second mode assumes negative values. There are also feeble oscillations in the damping factor pertaining to the first mode. Whenever the damping factor associated with the second mode increases, the damping factor associated with the first mode decreases and vice versa. This observation suggests energy exchange between the two modes. Revisiting Fig. 1 and the associated analysis, we find that the oscillations in the modal amplitudes are to be expected as a consequence of the energy exchange between the first and second modes of the structure. The free decay of the first and second modes is governed by both the damping and the two-to-one internal resonance. The results from ERA essentially yield equivalent linear damping factors which oscillate and assume negative values as a result of the nonlinear interactions in the structure.

IV. Discussion

Strictly speaking, ERA is intended for the identification of linear systems. However, one usually does not know the nature of a system a priori. The ERA method can be used in the first step towards its identification. If one identifies negative damping coefficients, it is not necessarily an indication of inaccurate modal extraction, but possibly an indication of nonlinear interactions. A sliding-window analysis can then confirm this indication and possibly point out the interacting modes. So, by examining the ERA results, it may be possible to identify the nonlinearly coupled modes. However, the identification of these modes may be limited by the complexity of a structure, noise levels in a test setup, etc. For the identified coupled structural modes, models such as those described in Sec. II may be necessary. Once a model has been chosen, a resonance-based approach[10] may be used for parametric identification, and hence, quantification of nonlinearities.

In Fig. 6, results obtained from an ERA sliding window analysis in experiments with the Mini-Mast structure[5] [Fig. 18(a) of Ref. 5] are provided. This figure is qualitatively similar to Fig. 5. The points corresponding to modes 1, 2, and 3, are, respectively, labeled 1, 2, and 3 in this figure. We see oscillations in the identified damping factors. The damping factors associated with all three modes assume negative values. Further, it is also observed that as the damping factor associated with one of the modes increases those associated with the other two modes decrease and vice versa. Oscillations and scatter are also seen in some of the other plots provided in Ref. 5. In the case of the Mini-Mast structure, there is a one-to-one frequency relationship between the first and second modes (modes 1B-X and 1B-Y of Ref. 5). There might be a nonlinear interaction between these two modes, as in the case of the cubically coupled oscillators of Sec. II. We suspect that the negative damping identified for some of the modes is a consequence of

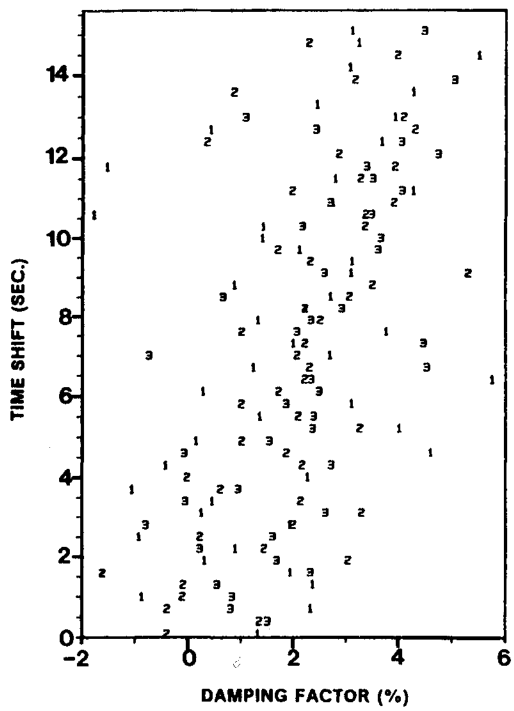


Fig. 6 ERA sliding-window results for the Mini-Mast structure reported in Ref. 5.

nonlinear interactions. Further, in other experiments with the Mini-Mast structure, in plots similar to Fig. 6, the scatter was seen to increase as the number of shakers used to excite the structure was increased. The following explanation is offered for this observation. As the number of shakers is increased, the number of excited structural modes is increased, resulting in an increase in the nonlinear interactions between the different structural modes. These interactions cause a scatter in the results obtained from the linear identification scheme.

In some recent experiments with a long and slender cantilever beam,[17] we observed nonlinear interactions in which energy was transferred from high-frequency to low-frequency modes. Band-limited random excitations were used to phenomenologically identify the nonlinear interactions in the beam. Further, ERA analysis for free-oscillation data from this cantilever beam yields oscillatory damping coefficients.[18] Nonlinear interactions observed in the beam may occur in flexible structures whose modal frequencies are spread over a wide frequency range (see, e.g., Ref. 5).

V. Concluding Remarks

In this work we considered phenomenological identification of nonlinear interactions in structures using linear identification techniques. In the presence of nonlinear interactions, linear system identification schemes yield damping factors that oscillate in time and sometimes assume negative values. The oscillations are distinct in character. When the damping factor associated with a particular mode increases those associated with the coupled modes decrease and vice versa. Results from linear identification techniques, such as the Eigenvalue Realization Algorithm, may be used to identify the nonlinearly coupled structural modes. However, this approach may be limited by factors such as the complexity of a structure and needs further investigation.

Acknowledgments

The work of the first two authors was supported by the Army Research Office under Grant DAAL03-89-K-0180 and the Office of Naval Research under Contract N00014-90-J-1149. We thank Cheri E. Tanner of NASA Langley Research Center and the reviewers for providing helpful suggestions.

References

[1]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations*, Wiley-Interscience, New York, 1979.

[2]Nayfeh, A. H., and Balachandran, B., "Modal Interactions in Dynamical and Structural Systems," *Applied Mechanics Reviews*, Vol. 42, No. 11, 1989, pp. 175–202.

[3]Juang, J.-N., and Pappa, R. S., "An Eigensystem Realization Algorithm for Modal Parameter Identification and Model Reduction," *Journal of Guidance Control, and Dynamics*, Vol. 8, No. 5, 1985, pp. 620–627.

[4]Pappa, R. S., and Juang, J.-N., "Some Experiences with the Eigensystem Realization Algorithm," *Sound and Vibration*, Vol. 22, 1988, pp. 30–34.

[5]Pappa, R. S., Schenk, A., and Noll, C., "ERA Modal Identification Experiences with Mini-Mast," 2nd USAF/NASA Workshop on Identification and Health Monitoring of Precision Space Structures, Pasadena, CA, March 27–29, 1990.

[6]Foster, C. L., Tinker, M. L., Nurre, G. S., and Till, W. A., "The Solar Array-Induced Disturbance of the Hubble Space Telescope Pointing System," NASA Marshall Space Flight Center Paper, 1990.

[7]Hanagud, S. V., Meyappa, M., and Craig, J. I., "Method of Multiple Scales and Identification of Nonlinear Structural Dynamic Systems," *AIAA Journal*, Vol. 23, No. 5, 1985, pp. 802–807.

[8]Mook, D. J., "Estimation and Identification of Nonlinear Dynamic Systems," *AIAA Journal*, Vol. 27, No. 7, 1989, pp. 968–974.

[9]Yasuda, K., Kawamura, S., and Watanabe, K., "Identification of Nonlinear Multi-Degree-of-Freedom Systems (Presentation of an Identification Technique)," *JSME International Journal*, Series III, Vol. 31, 1988, pp. 8–14.

[10]Nayfeh, A. H., "Parametric Identification of Nonlinear Dynamic Systems," *Computers & Structures*, Vol. 20, Nos. 1–3, 1985, pp. 487–493.

[11]Zavodney, L. D., and Hollkamp, J. J., "Experimental Identification of Internally Resonant Nonlinear Systems Possessing Quadratic Nonlinearity," *Proceedings of the AIAA 32nd Structures, Structural Dynamics, and Materials Conference* (Baltimore, MD), Pt. 4, AIAA Washington, DC, 1991, pp. 2755–2765 (AIAA Paper 91-1226).

[12]Haddow, A. G., Barr, A. D. S., and Mook, D. T., "Theoretical and Experimental Study of Modal Interaction in a Two-Degree-of-Freedom Structure," *Journal of Sound and Vibration*, Vol. 97, 1984, pp. 451–473.

[13]Nayfeh, A. H., and Zavodney, L. D., "Experimental Observation of Amplitude- and Phase-Modulated Responses of Two Internally Coupled Oscillators to a Harmonic Excitation," *Journal of Applied Mechanics*, Vol. 55, Sept. 1988, pp. 706–711.

[14]Balachandran, B., and Nayfeh, A. H., "Nonlinear Motions of a Beam-Mass Structure," *Nonlinear Dynamics*, Vol. 1, No. 1, 1990, pp. 39–61.

[15]Balachandran, B., and Nayfeh, A. H., "Observations of Modal Interactions in Resonantly Forced Beam-Mass Structures," *Nonlinear Dynamics*, Vol. 2, No. 2, 1991, pp. 77–117.

[16]Phan, M., Horta, L. G., Juang, J.-N., and Longman, R. W., "Linear System Identification via an Asymptotically Stable Observer," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (New Orleans, LA), AIAA, Washington, DC, 1991, pp. 1180–1194.

[17]Anderson, T. J., Balachandran, B., and Nayfeh, A. H., "Observations of Nonlinear Interactions in a Flexible Cantilever Beam," *Proceedings of the AIAA 33rd Structures, Structural Dynamics, and Materials Conference* (Dallas, TX), Pt. 3, AIAA, Washington, DC, 1992, pp. 1678–1686 (AIAA Paper 92-2332).

[18]Smith, S. W., Balachandran, B., and Nayfeh, A. H., "Nonlinear Interactions and the Hubble Space Telescope," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Hilton Head, SC), AIAA, Washington, DC, 1992 (AIAA Paper 92-4617).